From Description to Proof

CORNELIS DE GROOT

That sounds impossible. . . I would like to see such a rectangle. . . .

Do you agree that this line is inside the circle? Now look how I fold it over. . . .

You see, it is not on the rim, so a diameter is not an arc.

THESE QUOTES CAME FROM SIXTH-grade students during several geometry lessons. Because of my interest in how students stimulate one another's reasoning in mathematics, I spent some time in classrooms with teachers who were willing to step back and let their students express their thinking, and whose careful modeling would resonate in the students' reasoning at a later time. The two vignettes that follow illustrate the fact that student-to-student discourse and careful teacher modeling support a transition path to more formal mathematical reasoning.

Vignette 1: Classifying Quadrilaterals

A SERIES OF LESSONS ON CLASSIFYING QUADRILATERALS offered the following definition: "A rectangle is a parallelogram with at least one right angle."

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Mindy stood up and said, "I want to see such a rectangle!" She apparently thought that this rectangle would have only one right angle and the others would not be right angles. How could such a rectangle exist? At first, the teacher, Ms. Weiss (a pseudonym), replied that this statement required proof, which the students would learn to do in high school. The class was silent, and Ms. Weiss and Mindy exchanged glances. Ms. Weiss asked whether other students understood the question and whether they were interested in figuring out whether the statement was impossible. The class responded with interest.

Ms. Weiss took a piece of regular paper and asked the students whether they thought it was a rectangle. "Yes!" they said in chorus. She asked them how they were so sure, then proposed that they find out together. She said, "We are told in the statement that one of the corners was a right angle. I will put a dot in one of the corners" (see **fig. 1a**). She showed the paper to the class, then she folded the paper over (see **fig. 1b**). She asked the students what they could conclude about the covered angle, and all agreed that it was a right angle also. It had to be because it matched up with the first right angle. Then Ms. Weiss asked how she could show that the two remaining corners were also right angles. The students said, "Fold it the other way." Ms. Weiss folded the paper again, and the angles were congruent again (see **fig. 1c**). Mindy sat down convinced that she had asked an important question.

Seeking the level of the student

Initially, Ms. Weiss responded to Mindy's question on a theoretical level, saying that the statement required proof, which the students would learn about later. She explained to me that she could not immediately find a way to bridge instruction to Mindy's thinking. She imagined a proof using the definition of a rectangle and properties of parallel lines and believed that this exercise was inappropriate for a class with many students who were not ready for formal proof. Ms. Weiss said that she was stimulated by Mindy's persistent gaze and reflected on what she could do. She asked whether the class shared this burning perception of impossibility, then modeled the reasoning for the truth of the statement with a piece of regular paper.

Ms. Weiss focused on the rectangle in her modeling and did not actually prove the original statement. She returned to the original statement later in the sequence of lessons. Putting the statement aside for the moment did not present a problem because the teacher's goal was to help the students, through modeling, make sense of a statement that initially evoked an impossible mental image. The students' thinking was, in essence, "A rectangle cannot have only one right angle. It must have four right angles," and, possibly, "A parallelogram does not have right angles." The classroom discussion focused on the first interpretation. The students may have stumbled when they encountered the phrase "at least" or even ignored its meaning. Note that in the vignette, Ms. Weiss attempted to take the perspective of the student; that is, she started from a more literal response to the statement as a point of departure.

"I want to see such a rectangle"

In the first vignette, Ms. Weiss "found the level of the student" in her attempt to explain a statement that requires the student to think at either the descriptive or theoretical level rather than the visual level (van Hiele 1986). The visual approach used in this vignette allowed all students to match their internal visualizations to an external model. Ms. Weiss's demonstration supported the students in reconciling their mental images, evoked by taking a statement literally, with the physical reality of the model. According to van Hiele (1986), we assist in this reconciliation by seeking the level of the student. The teacher's modeling facilitated this process. After the modeling, students began to "see" the reasonableness of the statement that seemed to mismatch their initial literal thinking.

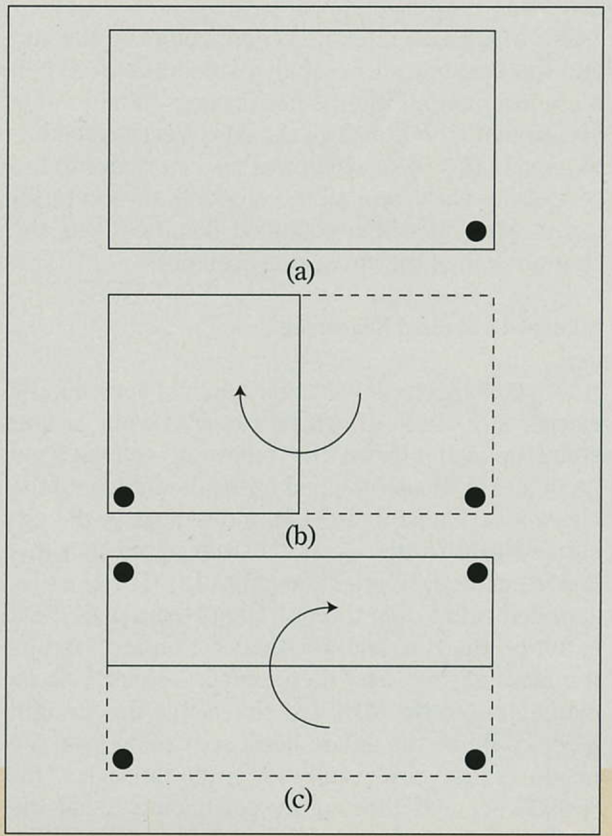

Fig. 1 A rectangle is a parallelogram with at least one right angle.

Vignette 2: Naming Arcs

A FEW WEEKS LATER, ONE STUDENT, NADIA, USED a similar type of modeling in reasoning with another student, Monique, who looked at a visual image in an unexpected way. In what started out as a simple five-minute review for an upcoming quiz, students' questions again prompted a different direction for the lesson. The students had completed an activity sheet for homework that reviewed terms related to a circle, for example, *radius, diameter, central angle, arcs,* and *chord.* In addition, they were to name these elements appropriately in symbolic form.

In reviewing the activity sheet, Vincent asked whether naming an arc *ABCD* would be acceptable (see **fig. 2**). Ms. Weiss remarked that arcs are not usually named in this way; it was, perhaps, a little long. A discussion followed in which the class reached consensus on a naming convention. Valerie asked whether a full circle is an arc, and if so, how it would be named. Quite a few students thought that a circle is not an arc. Ms. Weiss encouraged the students to try to name the arc of a full circle, and they realized that the arc's name would start and end with the same point, for example, $\overset{\frown}{ABA}$.

Next Joan asked whether a semicircle is an arc. Ms. Weiss responded by putting a drawing on the board (see **fig. 3**). The class discussion eventually focused on the fact that a semicircle is a part of the

"rim." Ms. Weiss identified some points on the arc, and the students successfully named it $\overset{\frown}{ACB}$. Then a student had an interesting insight. Monique observed that $\overset{\frown}{ASB}$ is also an arc. Ms. Weiss asked her to explain this idea, which was also attractive to the rest of the class, and all the students turned to listen to Monique. She explained that $\overset{\frown}{ASB}$ was the "bottom half of the circle squeezed up."

Student-to-Student Reasoning

After some discussion, Nadia, who had been quietly staring at a circle drawn on paper in front of her, stood up and offered the following counterargument as Ms. Weiss stepped out of the middle of the classroom. Nadia had drawn a diameter in the circle and said to the class, "Do you agree that this line is inside the circle?" (see **fig. 4a**). The class responded with a choral "yes." Next Nadia said, "So it is not on the rim, and it is then not an arc!" Again, the class agreed. She then folded the circle on its diameter (see **fig. 4b**) and stated that the straight piece could not be an arc because it came from the inside of the circle and not from the rim. This "proof" seemed thoroughly convincing to all the students in the class. Nadia sat down, pleased that she had contributed to the discussion.

Ms. Weiss's lesson plan had called for a quick review of the circle and its elements, but she let go of her plan as the need for discussion arose. In naming a circle as an arc, she stimulated the students to use the naming convention on which they had agreed. She helped the students link the visual model with symbolic rotation. The closed nature of the circle was slightly problematic, but the problem was resolved through symbolic consistency by naming the arc of the circle $\overset{\frown}{ABA}$. In the quest to find answers to their own questions, the students were making sense of the symbols, a process that in turn stimulated their exploratory thinking. When the question of whether a semicircle is an arc arose, Ms. Weiss presented a different definition of a semicircle than the one the students used. She put the drawing shown in **figure 3** on the board and reviewed rim and diameter and the area between them as composing a semicircle, which is an incorrect definition. In contrast, the students thought of the semicircle as the rim only. Ms. Weiss's depiction elicited a response from Monique, who saw the diameter as being part of the rim. Ms. Weiss did not give an answer to Monique's question but focused the class's attention on the rim. This emphasis appeared to be helpful when Nadia gave an explanation for the idea that the diameter is not an arc.

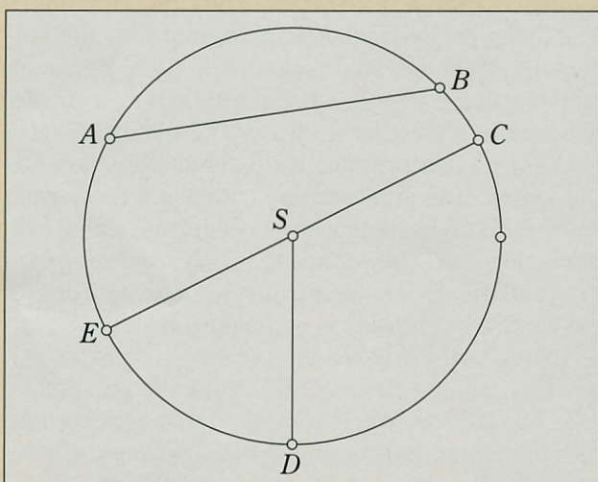

Fig. 2 Naming arcs

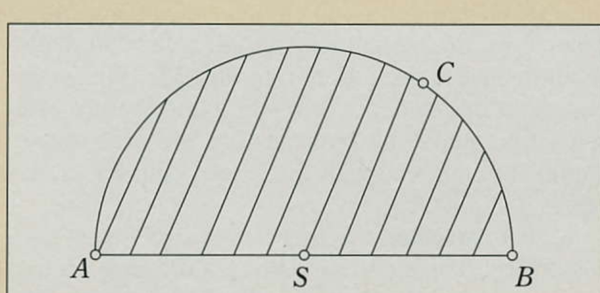

Fig. 3 Ms. Weiss's semicircle

Stepping out and stepping in: teenager talk

Here, the teacher deliberately chose when to step in and when to step out of the class discussion. She was constantly seeking the level of the students. The work on naming arcs led to an inference about the diameter. The students did not use the argument that the diameter could never be an arc because the line was straight and not curved. The initial topological statement seemed to have a degree of reasonableness for them. Nadia, who gave the proof, was modeling how the students could decide on their definition. Her modeling was similar to that offered by Ms. Weiss in the first vignette. The discussion held no sense of right or wrong but one of reasonableness. The representation of various levels of reasoning in the classroom was essential for this result. Both Monique and Nadia, and the rest of the class, stimulated deeper reasoning in one another while they tried to reconcile different views. Monique, who appeared to operate at the concrete-visual level, and Nadia, who seemed to reason more deductively, demonstrated that enormous leaps can be made by students at both levels when they are involved in discourse. Ms. Weiss had a different role in this vignette. She intervened very little and kept the whole class involved by posing prompts for thinking. Her diagram, although incorrect, served as such a prompt.

Levels of Thinking

INSTRUCTION IN THE TRANSITION YEARS SHOULD focus on connecting informal mathematical explorations of students to more sophisticated and complex high school mathematics (Hirsch and Lappan 1989, p. 614). Hirsch and Lappan (1989) identify three important dimensions along which change must occur: (1) from number to variable, (2) from specifics to generalization, and (3) from description to proof. I have focused on the third dimension, from description to proof. What understanding enables learners to make such a transition is an important question for educators. Theories have focused on linguistic development, the ability to think metaphorically, and the use of instructional devices. Taken together, these theories may help us understand students' thinking in this sixth-grade classroom over a period of time.

When considering the "I-see-it-so" explanations of developmentally younger children, Freudenthal (1978, p. 276) asserts that a change occurs in children at about the age of eleven or twelve: "Then something changes, children do not see anymore what they saw before." In elementary and middle school mathematics, teachers are frequently amazed that students report correct results time and time again but cannot communicate any thought process other than, "I just see it." Freudenthal states, "We should not be amazed about children's seeing-it-so, but rather about ourselves who do not see it so, or who judge that seeing-it-so is not enough, or believe it to be a miracle." Freudenthal also says that seeing-it-so prevents the child from making any verbal efforts (Freudenthal 1978, pp. 275–76). He believes that the main reason that students do not see-it-so at about eleven or twelve years of age is strongly related to their developing verbal abilities, which in turn suppress intuition.

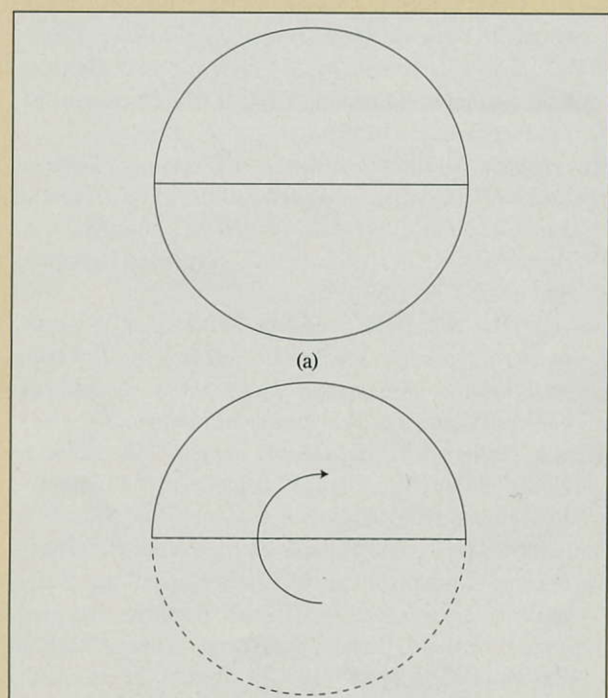

Fig. 4 A diameter is not an arc.

In the first vignette, a statement about the relationship between a rectangle and a parallelogram conflicted with Mindy's seeing-it-so; that is, a rectangle must have four right angles, whereas a parallelogram has none. The syntax of the statement also appeared to contribute to Mindy's conflict. Teachers who attempt to support their students in the transition to more formal reasoning often perceive that the students are hearing a foreign language. Van Hiele (1986) was troubled by this lack of communication and struggled with developing formal reasoning in middle school geometry classrooms. He initially identified three developmental levels at which students operate—the visual, the descriptive, and the theoretical. At the visual level, students view objects globally and appeal to intuitive forms of thinking, such as I-see-it-so. At the descriptive level, an object is seen as a structure of properties of that object. For example, in the second vignette, the students found that a diameter is not an arc. At the theoretical level, students apply deductive reasoning about relationships between objects. Although his work has Piagetian influences, van Hiele asserts that these developmental phases are discrete and do not follow one another naturally through accommodation and assimilation but, rather, require an instructional phase to get the student to the next level. For a detailed discussion, see the work of Fuys and Geddes (1984) and Fuys, Geddes, and Tischler (1984, 1988).

Students cannot explain, other than "I just see it"

Linguistic development is important for moving toward formal mathematical reasoning. According to van Dormolen (1991), students start to think more descriptively as they start to think metaphorically. Initially, children's confusion with teacher-speak relates to their difficulty in separating figurative speech from literal speech: "I want to see such

Transitional learners pop in and out of van Hiele levels

a rectangle." In this literal stage, the learner is bound by examples of a phenomenon as a concrete object: "It's the bottom half of the circle pushed up." When the learner starts to see these examples as metaphors, a transition is initiated. For example, a child might say, "A rectangle is like a parallelogram." Van Dormolen also contends that when students become further detached from the literal stage, as all examples are in a sense the same, metaphors become dead metaphors; that is, the meanings attain a natural place in the child's schema. For example, a child might then say, "A rectangle is a parallelogram." In the classroom, the transition from I-see-it-so to a mathematical understanding of a given situation can be facilitated by devices that make mental images externally visible, as demonstrated in the vignettes. Prompting students to think about and model mathematical ideas is a profitable instructional strategy. Asking one student to explain an idea to others who do not see-it-so and asking children to draw, model, or show what they see are also valuable approaches (Freudenthal 1978).

Concluding Remarks

IN MY OBSERVATIONS, I HAVE NOTICED THAT transitional learners pop in and out of van Hiele levels of reasoning, similar to excited electrons in an atom. Furthermore, these levels are distributed among the children in a class. Every day, opportunities exist for honing reasoning within the child and among the children.

Whereas Lappan and Hirsch speak of a transition from description to proof through exploration, van Hiele speaks about transitions from visual to descriptive to theoretical through deliberate instructional phases. Van Dormolen identifies a development that moves from examples to metaphors to dead metaphors. In these transition models, the middle phase is still grounded in the examples of the phenomenon, whereas the theoretical level has become detached from the concrete reality and has its own life, so to speak. The students move from thinking that "a rectangle is like a parallelogram" to thinking that "a rectangle is a parallelogram."

A few words of caution are in order. The word *levels* contains a strong element of bias. We tend to rank levels in order and value higher levels over lower ones. Taking such a stance may lead to undervaluing visual and descriptive reasoning. I suggest that valuing all levels is important in the middle grades to support the development of mathematical reasoning.

Through careful and deliberate instruction, as demonstrated in the vignettes, we can take advantage of the naturally rich and different ways of thinking that all students in the classroom bring to the discourse. Through discourse, we foster the development of reasoning. Hirsch and Lappan provide us with three fundamental domains in which the transition from informal to formal reasoning "must" take place. I have illustrated some frameworks that we can use to start thinking about how to implement their research on transitions in the classroom. Much research is needed in each of these domains of transition, as are studies of effective instructional techniques. I suggest that classroom discourse is the main vehicle because it intrinsically values the understandings that all children bring to the classroom.

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